



Integrating Fuzzy Logic and Chaos

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Dedicated to Professor Janusz Kacprzyk for his 60th birthday

Abstract. Motivated by the current studies on the interactions between fuzzy logic and chaos theory, for instance, fuzzy modeling of chaotic systems using Takagi-Sugeno (TS) models, linguistic descriptions of chaotic systems, fuzzy control of chaos, complex fuzzy systems, and a combination of fuzzy control technology and chaos theory for an engineering practice, this survey paper aims to provide some heuristic research achievements and insightful ideas to attract more attention on the topic, interactions or relationship between fuzzy logic and chaos theory, which are related at least within the context of human reasoning and information processing.

1 Introduction

The naissance of fuzzy logic and the prosperity of scientific research on chaos theory occurred almost at the same time in the 1960s, a decade full of confusion, when scientists faced difficulties in dealing with imprecise information and complex dynamics. A set theory and then an infinite-valued logic of Lotfi A. Zadeh were so confusing that they were called fuzzy set theory and fuzzy logic in 1965; a deterministic system found by Edward N. Lorenz in 1963 to have random behaviors was so unusual that it was lately named a chaotic system [1, 2].

Since then, fuzzy set theory and chaos theory have independently developed along their own ways and matured as sciences (although still evolving). They have provided many insights into previously intractable and inherently imprecise or complex nonlinear natural phenomena.

In particular, fuzzy systems technology has achieved its maturity with widespread applications in many industrial, commercial and technical fields, ranging from control, automation, and artificial intelligence to image/signal processing, pattern recognition, and electronic commerce. Chaos, on the other hand, was considered as one of the three monumental discoveries of the twentieth century together with the theory of relativity and quantum mechanics. As a very special nonlinear dynamical phenomenon, chaos has reached its current outstanding status from being merely a scientific curiosity in the mid-1960s to an applicable technology in the late 1990s.

Why do we bring together the two seemingly unrelated concepts to study their interactions and relationships? On the one hand, finding the intrinsic relationship between fuzzy logic and chaos theory is certainly of significant interest and of potential importance. The past twenty years have indeed witnessed some serious explorations of the interactions between fuzzy logic and chaos theory, leading to such research topics as fuzzy modeling of chaotic systems using Takagi-Sugeno models [3], linguistic descriptions of chaotic systems [4, 5], fuzzy control of chaos [6], and a combination of fuzzy control technology and chaos theory for various engineering practices [7]. On the other hand, the reason to study the interactions between fuzzy logic and chaos theory lies in that they are related at least within the context of human reasoning and information processing. In fact, fuzzy logic resembles human approximate reasoning using imprecise and incomplete information with inaccurate and even self-conflicting data to generate reasonable decisions under such uncertain environments, while chaotic dynamics play a key role in human brains for processing massive amounts of information instantly. It is believed that the capability of humans in controlling chaotic dynamics in their brains is more than just an accidental by-product of the brain's complexity, but rather, it could be the chief property that makes the human brain different from any artificial-intelligence machines [8]. It is also believed that to understand the complex information processing within the human brain, fuzzy data and fuzzy logical inference are essential, since precise mathematical descriptions of such models and processes are clearly out of question with today's scientific knowledge.

What is necessary to mention is that Lotfi A. Zadeh has integrated fuzzy logic and chaos theory into the concept of *soft computing* (SC), where he states that SC consists of fuzzy logic (FL), neural network theory (NN) and probabilistic reasoning (PR), with the latter subsuming parts of belief networks, genetic algorithms, chaos theory and learning theory. It is noted that SC is not a melange of FL, NN and PR. Rather, it is an integration in which each of the partners contributes a distinct methodology for addressing problems in their common domain. In this perspective, the principal contributions of FL, NN and PR are complementary rather than competitive.

This paper aims to provide some heuristic research achievements and insightful ideas to attract more attention on the topic, through reviewing the current studies on the interactions between fuzzy logic and chaos theory, including fuzzy modeling of chaotic systems using Takagi-Sugeno (TS) models, linguistic descriptions of chaotic systems, fuzzy control of chaos, complex fuzzy systems, and a combination of fuzzy control technology and chaos theory for an engineering practice.

2 Fuzzy Definition of Chaos

The term *chaos* associated to an interval map was first formally introduced into mathematics by Li and Yorke in 1975 [9], where they established a simple

criterion for the existence of chaos in one-dimensional difference equations, i.e., “period three implies chaos”.

However, a definitive, universally accepted, and completely rigorous mathematical definition of chaos is not yet available in the scientific literature to provide a fundamental basis for studying such exotic phenomena. Instead, various alternative, but closely related definitions of chaos have been proposed, along with mechanisms giving rise to such behavior. Among those, the original definition of Li and Yorke and its fine tuning by Devaney seem to be the most popular. These definitions of chaos have been generated to difference equations in \mathcal{R}^n , Banach spaces and complete metric spaces. In particular, an application of the definition in complete metric spaces to chaotic dynamics on the metric space (\mathcal{E}^n, D) of fuzzy sets on the base space \mathcal{R}^n may lay a foundation for further study on the interactions between fuzzy logic and chaos theory [10].

It is shown by the result of Li and Yorke that such chaotic behavior could arise in quite simple systems and could be generated by quite simple mechanisms. In particular, the “period three implies chaos” result in scalar difference equations involves *noninvertible*, *continuous* maps rather than more demanding diffeomorphisms. Consequently, one may ask to what extent the one-dimensional result of Li and Yorke carries over to a higher dimensional difference equation

$$x_{k+1} = f(x_k), \quad k = 0, 1, 2, \dots, \quad (1)$$

with a continuous map $f : X \rightarrow X$, where X is a closed subset of \mathcal{R}^n for $n \geq 2$.

A counterexample shows that the result does not carry over to higher dimensions without some suitable restriction on the class of maps f [11]. To determine a suitable class of maps for which the result of the Li and Yorke might hold in higher dimensions, we found that the one-dimensional maps for which the difference equation (1) have cycles of period three all have graphs with a hump, i.e., which fold over on themselves, namely, they are not one-to-one maps. This suggests that attention might profitably be restricted to maps that are not one-to-one. This was done by Marotto [12] who showed that difference equations on \mathcal{R}^n defined in terms of continuously differentiable maps with *snap-back repellers*, so consequently not one-to-one, behave chaotically in the sense of Li and Yorke. His proof used the inverse function theorem for one-to-one local restrictions of the maps and the Brouwer fixed point theorem, but otherwise paralleled the proof of Li and Yorke for one-dimensional maps. This was extended in 1981 to maps with a saddle point by Kloeden [13] and Shiraiwa and Kurata [14]. Further, the result of Kloeden for the case of saddle points in a finite dimensional Euclidean space \mathcal{R}^n can be easily extended by using the Schauder fixed point theorem to a Banach space. Even more generally, some criteria for chaos of difference equations in general complete metric spaces have been given in [15].

What we concern here is to generalize the Li-Yorke and Marotto definitions to be applicable to maps from a space of fuzzy sets into itself, namely the metric space (\mathcal{E}^n, D) of fuzzy sets on the base space \mathcal{R}^n . The result to be given below is essentially an adaptation of the result in a Banach space, which is possible because the metric spaces of fuzzy sets under consideration can be embedded as a cone in a certain Banach space.

Here, we simply give the Kaleva fixed point theorem without giving some basic terminologies [16].

Theorem 1. (Kaleva)

Let $f : \mathcal{E}^n \rightarrow \mathcal{E}^n$ be continuous and let \mathcal{X} be a non-empty compact convex subset of \mathcal{E}^n such that $f(\mathcal{X}) \subseteq \mathcal{X}$. Then f has a fixed point $\bar{u} = f(\bar{u}) \in \mathcal{X}$.

Consider an iterative scheme of fuzzy sets

$$u_{k+1} = f(u_k), \quad k = 1, 2, \dots, \quad (2)$$

where f is a continuous map from the space of fuzzy sets \mathcal{E}^n into itself. Using the Kaleva fixed point theorem, sufficient conditions will be given below for a map on fuzzy sets to be chaotic.

Theorem 2. (Kloeden [17])

Let $f : \mathcal{E}^n \rightarrow \mathcal{E}^n$ be continuous and suppose that there exist non-empty compact subsets \mathcal{A} and \mathcal{B} of \mathcal{E}^n , and integers $n_1, n_2 \geq 1$ such that

- (i) \mathcal{A} is homeomorphic to a convex subset of \mathcal{E}^n ,
- (ii) $\mathcal{A} \subseteq f(\mathcal{A})$,
- (iii) f is expanding on \mathcal{A} , that there exists a constant $\lambda > 1$ such that

$$\lambda D(u, v) \leq D(f(u), f(v))$$

for all $u, v \in \mathcal{A}$,

- (iv) $\mathcal{B} \subset \mathcal{A}$,
- (v) $f^{n_1}(\mathcal{B}) \cap \mathcal{A} = \emptyset$,
- (vi) $\mathcal{A} \subseteq f^{n_1+n_2}(\mathcal{B})$,
- (vii) $f^{n_1+n_2}$ is one-to-one on \mathcal{B} .

Then the map f is chaotic.

It is noted that difference equations generated by Poincaré section maps provide a link between the dynamics of discrete-time dynamical systems and continuous-time dynamical systems. However, it has been much more difficult to give a mathematically rigorous proof of the existence of chaos in a continuous-time nonlinear autonomous systems. Even one of the classic icons of modern nonlinear dynamics, the Lorenz attractor, now known for 40 years, was not proven rigorously to be chaotic until 1999 by Warwick Tucker of the University of Uppsala in his Ph.D. dissertation [18]. A commonly agreed analytic criterion for proving the existence of chaos in continuous-time systems is based on the fundamental work of Shil'nikov, known as the Shil'nikov method or Shil'nikov criterion [19], whose role is in some sense equivalent to that of the Li-Yorke definition in the discrete setting. The Shil'nikov criterion guarantees that complex dynamics will occur near homoclinicity or heteroclinicity when an inequality (Shil'nikov inequality) is satisfied between the eigenvalues of the linearized flow around the saddle point(s), i.e., if the real eigenvalue is larger in modulus than the real part of the complex eigenvalue. Complex behavior always occurs when the saddle set is a limit cycle.

3 From Chaos to Fuzziness

This section discusses fuzzy modeling of chaotic systems, meaning to transform chaotic systems into fuzzy formulations.

Fuzzy system models basically fall into two categories, which differ fundamentally in their abilities to represent different types of information. The first category includes linguistic models, which have been referred to so far as Mamdani fuzzy models. They are based on collections of IF-THEN rules with vague predicates and use fuzzy reasoning [20, 21]. In these models, fuzzy quantities are associated with linguistic labels, and a fuzzy model is essentially a qualitative expression of the underlying system. Models of this type form a basis for qualitative modeling that describes the system behavior by using natural language [22]. A corresponding fuzzy logic controller is a prototypical example of such a linguistic model, in which its rules give a linguistic expression of the control strategy in a common sense.

The second category of fuzzy models is based on the Takagi-Sugeno (TS) method of reasoning [23–25]. These models are formed by logical rules that have a fuzzy antecedent part and a functional consequent. They are combinations of fuzzy and nonfuzzy models. Fuzzy models based on the TS method of reasoning integrate the ability of linguistic models for qualitative knowledge representation with great potential for expressing quantitative information.

3.1 Fuzzy Modeling of Chaotic Systems based on Mamdani Model

Fuzzy logic allows to model processes in a linguistic manner. The basic configuration of a fuzzy logic system is composed of a fuzzyfier, a fuzzy rule-base, a fuzzy inference engine, and a defuzzyfier, where the fuzzy rule-base consists of a collection of fuzzy IF-THEN rules, and the fuzzy inference engine uses these fuzzy IF-THEN rules to determine a map from fuzzy inputs to fuzzy outputs based on fuzzy composition rules.

A systematic approach for modeling chaotic systems using Mamdani model is never available, Baglio *et al.* [4, 5] have, however, subtly derived Mamdani fuzzy models of some typical chaotic systems. To do so, a good description of chaotic systems is required. The definition of chaotic behaviors involves the three fundamental concepts of transitivity, density of periodic orbits, and sensitivity to initial conditions [26]. Furthermore, from a qualitative point of view, chaos can be defined by monitoring the time evolution of trajectories emanating from nearby points on the attractor. In a chaotic system, points that are close to each other repel themselves so that the flow stretches. Then, a folding action must take place for the chaotic behavior to combine with the boundedness of the attractor. The stretching and folding features of the flow are responsible for the sensitivity to initial conditions, and characterize the chaotic behavior.

To develop a fuzzy model of the evolution of a chaotic signal x , two variables can be considered as inputs, i.e., the center value $x(k)$, which is the nominal value of the state x at the instant k , and the uncertainty $d(k)$ on the center value. In terms of fuzzy description, this means that the model contains four

linguistic variables, i.e., $x(k)$, $x(k + 1)$, $d(k)$ and $d(k + 1)$. The whole set of rules has to determine the values $x(k + 1)$ and $d(k + 1)$ from the values $x(k)$ and $d(k)$.

Take the logistic map $x(k+1) = \mu x(k)(1-x(k))$, $\mu = 4$, which shows a single-scroll attractor, as an example to illustrate the modeling procedure. In this single scroll system, x tends to move out from the trivial equilibrium point $x_1^* = 0$ until x begins to oscillate around the nontrivial equilibrium point $x_2^* = 3/4$. The increasing amplitude of the oscillations forces the trajectory to enter again the neighborhood of $x_1^* = 0$, where, due to its instability, the above process repeats. The linguistic variables of the system, $(x(k), x(k + 1), d(k), d(k + 1))$, take five linguistic values: zero (Z), small (S), medium (M), large (L) and very large (VL). The fuzzy sets associated to these linguistic values are shown in Fig. 1. They are constructed in such a way that the equilibrium point $x_2^* = 3/4$ is between the fuzzy set M and the fuzzy set L .

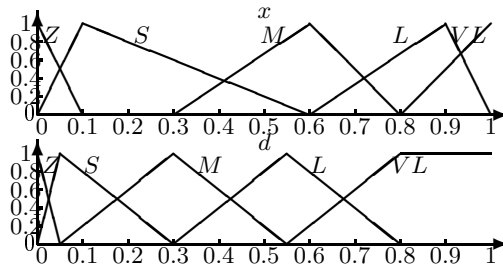


Fig. 1. The fuzzy sets for x (left) and d (right): logistic map

In other words, when x is smaller than the nontrivial equilibrium point $x_2^* = 3/4$, it tends to increase and, when x is very large, it tends to decrease, which can be summarized in the following rules.

$$R_1 : \text{ IF } x(k) \text{ is } S \text{ THEN } x(k + 1) \text{ is } M;$$

$$R_2 : \text{ IF } x(k) \text{ is } VL \text{ THEN } x(k + 1) \text{ is } Z.$$

In this way, a complete set of fuzzy rules to generate a single-scroll chaotic system is summarized in Table 1

The trajectory of the fuzzy system in the phase space is shown in Fig. 2 with the center-of-sums defuzzification method and the product as t -norm, which is obviously similar to that of the original logistic map.

3.2 Fuzzy Modeling of Chaotic Systems based on TS Model

As mentioned above, the TS fuzzy model adopts linear functions rather than fuzzy sets in the consequence part, thus, linearization methods are often used in modeling, where the following theorem can be used to convert the nonlinear terms in the nonlinear systems to weighted linear sums of some linear functions [27].

Table 1. Fuzzy rules implementing a single scroll chaotic system

$x(k)/d(k)$	Z	S	M	L	VL
Z	Z/Z	Z/M	Z/M	S/VL	L/L
S	M/Z	M/M	M/M	M/VL	L/S
M	L/Z	L/M	L/M	L/VL	VL/S
L	M/Z	M/M	M/M	M/VL	Z/S
VL	Z/Z	Z/M	Z/M	Z/VL	Z/L

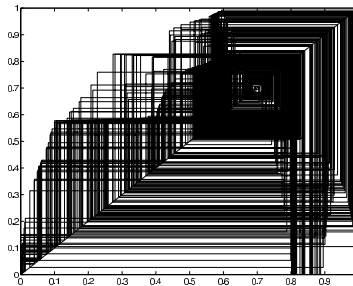


Fig. 2. The logistic map generated by the fuzzy model

Theorem 3. Consider the following nonlinear term:

$$f_n = x_1 x_2 \cdots x_n, \tag{3}$$

where $x_i \in [M_1^i, M_2^i]$. Formula (3) can exactly be represented by a linear weighted sum of the form

$$f_n = \left(\sum_{i_2, i_3, \dots, i_n=1}^2 \mu_{i_2 i_3 \dots i_n} \cdot g_{i_2 i_3 \dots i_n} \right) x_1, \tag{4}$$

where

$$g_{i_2 i_3 \dots i_n} = \prod_{j=2}^n M_{i_j}^j, \quad \mu_{i_2 i_3 \dots i_n} = \prod_{j=2}^n \Gamma_{i_j}^j,$$

in which $\Gamma_{i_j}^j$ is positive semi-definite for all $x_j \in [M_1, M_2]$, defined as follows:

$$\Gamma_1^j = \frac{-x_j + M_2^j}{M_2^j - M_1^j}, \quad \Gamma_2^j = \frac{x_j - M_1^j}{M_2^j - M_1^j}.$$

Proof. It can be easily proved using inductive reasoning, thus, the proof is omitted here.

Herewith, for most of chaotic systems, their exact TS fuzzy models can be easily derived to be with only two fuzzy rules. Here, the word “exact” means that the defuzzified output of the TS fuzzy models are mathematically identical to that of the original nonlinear systems.

For instance, the Lorenz equations,

$$\frac{d}{dt} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\sigma x + \sigma y \\ rx - y - xz \\ xy - bz \end{bmatrix}, \quad (5)$$

where $\sigma, r, b > 0$ are parameters, can be expressed as the following TS fuzzy model:

$$\begin{aligned} \text{Rule 1: IF } x(t) \text{ is about } M_1 \text{ THEN } \frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} &= A_1 \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \\ \text{Rule 2: IF } x(t) \text{ is about } M_2 \text{ THEN } \frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} &= A_2 \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \end{aligned}$$

where

$$A_1 = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & -M_1 \\ 0 & M_1 & -b \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & -M_2 \\ 0 & M_2 & -b \end{bmatrix},$$

and the membership functions are

$$\Gamma_1 = \frac{-x + M_2}{M_2 - M_1}, \quad \Gamma_2 = \frac{x - M_1}{M_2 - M_1},$$

where $\Gamma_i, i = 1, 2$, are positive semi-definite for all $x \in [M_1, M_2]$.

4 From Fuzziness to Chaos

It is clear to see that fuzzy modeling of chaotic systems implies that fuzzy systems can be also chaotic. In contrast to the last section, “from fuzziness to chaos” means to make originally stable or non-chaotic fuzzy systems chaotic. This is of practical significance, since chaos can actually be useful under certain circumstances, and there is growing interest in utilizing the very nature of chaos [28–31].

One simple, yet mathematically rigorous control method from the engineering feedback control approach was developed, where a linear state-feedback controller with a uniformly bounded control-gain sequence can be designed to make all Lyapunov exponents of the controlled system strictly positive and arbitrarily assigned. Moreover, such a controller can be designed for an arbitrarily given, n -dimensional dynamical system that could originally be nonchaotic or even asymptotically stable. The goal of chaotification is finally achieved with a simple modulus operation or a sawtooth (or even a sine) function. The design criterion is to use the definition of chaos given by Devaney or Li-Yorke [9],

while for the n -dimensional case the Marotto theorem [12] was used for a proof. For the continuous-time case, a general approach to make an arbitrarily given autonomous system chaotic has also been proposed recently [32–35]. Here, the main tool to use is time-delay feedback perturbation on a system parameter or as an exogenous input [33].

These chaotification techniques can be applied to TS fuzzy systems, where the so-called parallel distributed compensation (PDC) technique is employed to determine the structure of a fuzzy controller [38–43].

It is remarked that modeling of chaotic systems based on Mamdani models discussed above provides a means to generate chaos from Mamdani fuzzy systems in a linguistic manner.

5 Fuzzy Control of Fuzzy Chaotic Systems

For many years, the main feature of chaos, i.e., the extreme sensitivity to initial conditions, made chaos undesirable, and most experimentalists consider such characteristics as something to be strongly avoided [36, 37]. In addition to this feature, chaotic systems have two other important ones. First, there are infinite many unstable periodic orbits embedded in the underlying chaotic attractor, and second, the dynamics in the chaotic attractor is ergodic, which implies that during its temporal evolution the system ergodically visits any small neighborhood of every point in each of the unstable periodic orbits embedded within the chaotic attractor.

Owing to these properties, a fuzzy chaotic system, the fuzzy formulation of a chaotic system, can be seen as shadowing some periodic behavior at a given time, and erratically jumping from one to another periodic orbit. Thus, when a trajectory approaches ergodically a desired periodic orbit embedded in the chaotic attractor, one can apply small perturbations to stabilize such an orbit. Therefore, we can say that the extreme sensitivity of a chaotic system to changes in its initial conditions may be very desirable in practical experimental situations [37]. It suffices to note that, due to chaos, using the same chaotic system one is able to produce infinite many desired dynamical behaviors (either periodic or not periodic) only with properly chosen tiny perturbations. This property is not shared by non-chaotic systems, because the perturbations needed therein for producing a desired behavior must, in general, be of the same order of magnitude as the unperturbed dynamical variables.

Generally, chaos control approaches can be divided into two broad categories: feedback and nonfeedback (or say, open-loop) control approaches. Feedback control methods do not change the controlled systems and stabilize unstable periodic orbits embedded in chaotic attractors, while nonfeedback control methods slightly change the controlled system, mainly by a small tuning of control parameter, changing the system behavior from chaotic attractor to periodic orbit, which is close to the initial attractor.

It is known that the nonfeedback approach is much less flexible, and requires more prior knowledge of motion. To apply such an approach, one does

not have to follow the trajectory. The control can be activated at any time, and one can switch from one periodic orbit to another without returning into the chaotic behavior. This approach can be very useful in mechanical systems, where the feedback control systems are often very large (sometimes larger than the system controlled). The extremely simple, easily implementable, low-cost, and reliable nonfeedback approaches are widely applied in many physical experiments and industrial processes today, particularly for nonlinear dynamical systems, as a unified feedback control approach has not been fully established for general nonlinear dynamical systems. Roughly speaking, the nonfeedback approaches include the entrainment and migration control method [44–46], control through external forcing [47, 48], while the feedback approaches include the Ott-Grebogi-Yorke (OGY) method, engineering feedback control method, and Pyragas's time-delayed feedback control method [49–51]. In addition, chaos control approaches include conventional linear and nonlinear control, adaptive control, neural networks-based control, fuzzy control, and another very important topic, synchronization of chaos.

6 Fuzzy-Chaos-based Applications: An Example

Combination of fuzzy logic and chaos theory may provide a new means in engineering practice. An example is a fuzzy-model-based chaotic cryptosystem introduced in [7, 52–54].

Cryptography concerns the ways in which communications and data can be encoded to prevent disclosure of their contents through eavesdropping or message interception, using codes, ciphers, or other methods, so that only certain people can see the real messages. So far, varieties of cryptographic methods have been proposed to secure Internet communication. For instance, the Data Encryption Standard (DES) is adopted as a U.S. Federal Information Processing Standard for encrypting unclassified information. Others include IDEA (International Data Encryption Algorithm), and RSA (developed by Rivest, Shamir and Adleman). These encryption algorithms are based on number theory. However, none of them is absolutely secure. Therefore, some emerging theories, such as chaos theory, are always desirable to be adopted to strengthen existing cryptography. The reason of applying chaos theory in cryptography lies in its intrinsic essential properties, such as sensitivity to initial conditions (or control parameters) and ergodicity, which meet Shannon's requirements of confusion and diffusion for cryptography. In addition, chaotic signals are typically broadband, noise-like, and difficult to predict. Therefore, they can be used in various context for masking information-bearing waveforms. They can also be used as modulating waveforms in spread-spectrum systems. The idea of chaotic masking is to directly add the message in a noise-like chaotic signal at the end of the transmitter, while chaotic modulation is by injecting the message into a chaotic system as a spread-spectrum transmission. Later, at the receiver, a coherent detector with some signal processing is employed to recover the message. But the signal masking or parameter modulation approach to chaotic communication

only provides a lower level of security as stated in [55]. Using basic cryptosystem theory, a fuzzy-model-based chaotic cryptosystem has been proposed to provide a methodology with a higher level of security. There, Luré type discrete-time chaotic systems are first exactly represented by TS fuzzy models. Then, a superincreasing sequence is generated by using a chaotic signal, which can flexibly be used as an output of the TS fuzzy chaotic drive system, or any state in which the synchronization error approaches zero. In terms of a cryptosystem, the plaintext (message) is encrypted using the superincreasing sequence at the drive system side, which results in the ciphertext. The ciphertext may be added to the output or state of the drive system using the methodologies proposed in [7, 52]. Further, this the ciphertext embedding scalar signal is sent to the response system end. Following the design of a response system, the chaotic synchronization between a drive and a response system is achieved by solving LMIs. By the synchronization, one can regenerate the same superincreasing sequence and recover the ciphertext at the response system end. Finally, using the regenerated superincreasing sequence, the ciphertext is decrypted into the plaintext. The block diagram of the whole cryptosystem is shown in Fig. 3.

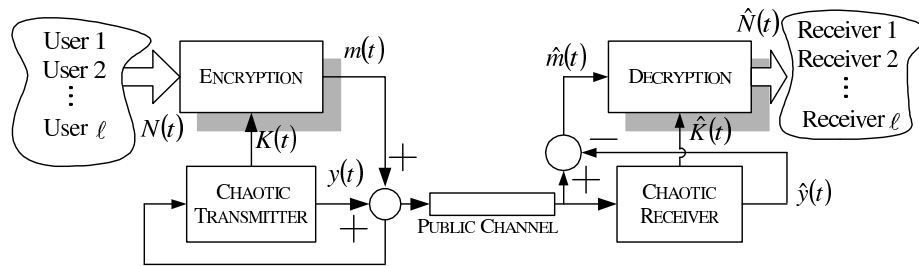


Fig. 3. Block diagram of chaotic encryption methodology

7 Conclusions

This paper has reviewed the current studies on the interactions between fuzzy logic and chaos theory, from the following aspects: fuzzy modeling of chaotic systems using Takagi-Sugeno models, linguistic descriptions of chaotic systems, fuzzy control of chaos, complex fuzzy systems, and a combination of fuzzy control technology and chaos theory for various engineering practices. What needs to emphasize is that in spite of the efforts on exploring the interactions between fuzzy logic and chaos theory, it is still far away from fully understanding their mutual relationships. Although this paper may not give insight into their relations or may raise more questions than it can provide answers, we hope that it nevertheless contains seeds for future brooming research.

References

1. Z. Li, W. Halang, and G. Chen, *Integration of Fuzzy Logic and Chaos Theory*, Springer-Verlag: Heidelberg, 2006.
2. Z. Li, *Fuzzy Chaotic Systems – Modeling, Control, and Applications*, Springer-Verlag: Heidelberg, 2006.
3. H.O. Wang, K. Kanaka and T. Ideda, Fuzzy modeling and control of chaotic systems, *Proc. 1996 IEEE Int. Symp. on Circ. Syst., ISCAS'96*, pp. 209–212.
4. S. Baglio, L. Fortuna and G. Manganaro, Design of fuzzy iterators to generate chaotic time series with assigned Lyapunov exponent, *Electronics Letters*, Vol. 32, No. 4, 1996, pp. 292–293.
5. M. Porto and P. Amato, A fuzzy approach for modeling chaotic dynamics with assigned properties, *The 9th IEEE Int. Conf. on Fuzzy Systems, 2000, FUZZ-IEEE 2000*, Vol. 1, pp. 435–440.
6. O. Calvo and J. L. Cartwright, Fuzzy control of chaos, *Int. J. Bifurc. Chaos*, Vol. 8, No. 8, 1998, pp. 1743–1747.
7. K.-Y. Lian, T.-S. Chiang, C.-S. Chiu and P. Liu, Synthesis of fuzzy model-based designs to synchronization and secure communications for chaotic systems, *IEEE Trans. Syst., Man, and Cybern. – Part B: Cybernetics*, Vol. 31, No. 1, pp. 66–83, 2001.
8. L. Glass and M. C. Mackey, *From Clocks To Chaos*, Princeton University Press, 1988.
9. T. Y. Li and J. A. Yorke, Period three implies chaos, *Amer. Math. Monthly*, Vol. 82, pp. 481–485, 1975.
10. P. Kloeden and Z. Li, Li-Yorke chaos in higher dimensions: a review, *Journal Difference Equations and Applications*, the special issue in Memory of Professor Bernd Aulbach, Vol. 12, No. 3–4, pp. 247–279, 2006.
11. P. E. Kloeden, Cycles and chaos in higher dimensional difference equations, Yu. A. Mitropolsky (Ed.), *Proc. 9th Int. Conf. Nonlinear Oscillations*, Vol. 2, pp.184–187, (Kiev Naukova Dumka) 1984.
12. F. R. Marotto, Snap-back repellers imply chaos in \mathbf{R}^n , *J. of Math. Anal. Appl.*, Vol.63, pp. 199–223, 1978.
13. P. E. Kloeden, Chaotic difference equations in \mathbf{R}^n , *J. Austral. Math. Soc. (Series A)*, Vol. 31, pp. 217–225, 1981.
14. K. Shiraiwa and M. Kurata A generalization of a theorem of Marroto, *Nagoya Math J.*, **82**, pp. 83–97, 1981.
15. Y. M. Shi and G. Chen, Chaos of discrete dynamical systems in complete metric spaces, *Chaos, Solitons and Fractals*, Vol. 22, pp. 555–571, Nov. 2004.
16. O. Kaleva, On the convergence of fuzzy sets, *Fuzzy Sets and Systems*, Vol. 17, pp. 53–65, 1985.
17. P. E. Kloeden, Chaotic iterations of fuzzy sets, *Fuzzy Sets and Systems*, Vol. 42, No. 1, pp. 37–42, 1991.
18. W. Tucker, The Lorenz attractor exists, *C. R. Acad. Sci. Paris*, Vol. 328, pp. 1197–1202, 1999.
19. C. P. Silva, Shil'nikov theorem—a tutorial, *IEEE Trans. Circuits Syst.-I*, Vol. 40, pp. 675–682, 1993.
20. R. M. Tong, The construction and evaluation of fuzzy models, in *Advances in Fuzzy Sets Theory and Applications*, M. M. Gupta, R. K. Ragade and R. R. Yager (eds.), North-Holland: Amsterdam, 1979, pp. 559–576.
21. W. Pedrycz, *Fuzzy Control and Fuzzy Systems*, Wiley: New York, 1989.

22. M. Sugeno and T. Yasukawa, A fuzzy-logic-based approach to qualitative modeling, *IEEE Trans. Fuzzy Systems*, Vol. 1, 1993, pp. 7–31.
23. T. Takagi and M. Sugeno, Derivation of fuzzy control rules from human operators actions, *Proc. IFAC Symposium on Fuzzy Information*, Marseille, 1983, pp. 55–60.
24. T. Takagi and M. Sugeno, Fuzzy identification of systems and its application to modeling and control, *IEEE Trans. Syst., Man, and Cyber.*, Vol. 15, pp. 116–132, 1985.
25. M. Sugeno and G. T. Kang, Fuzzy modeling and control of multilayer incinerator, *Fuzzy Sets and Systems*, Vol. 18, 1986, pp. 329–346.
26. H.-O. Peitgen, H. Jürgens and D. Saupe, *Chaos and fractals – New frontiers of science*, 2nd ed., Springer, 2004.
27. H. J. Lee, *Robust Fuzzy-Model-Based Controller Design for Nonlinear Systems with Parametric Uncertainties*, Master thesis, Yonsei University, Seoul, Korea, 2000.
28. G. Chen and X. Dong, *From Chaos to Order: Methodologies, Perspectives and Applications*, World Scientific: Singapore, 1998.
29. G. Chen, Chaos: Control and anticontrol, *IEEE Circuits Syst. Soc. Newsletter*, Vol. 9, No. 1, pp. 1–5, 1998.
30. S. J. Schiff, K. Jerger, D. H. Duong, T. Chang, M. L. Spano and W. L. Ditto, Controlling chaos in the brain, *Nature*, Vol. 370, pp. 615–620, 1994.
31. W. Yang, M. Ding, A. J. Mandell and E. Ott, Preserving chaos: Control strategies to preserve complex dynamics with potential relevance to biological disorders, *Phys. Rev. E*, Vol. 51, pp. 102–110, 1995.
32. X. F. Wang and G. Chen, Chaotifying a stable LTI system by tiny feedback control, *IEEE Trans. Circuits and Systems-I*, Vol. 47, No. 3, pp. 410–415, 2000.
33. X. F. Wang, G. Chen and X. Yu, Anticontrol of chaos in continuous-time systems via time-delay feedback, *Chaos*, Vol. 10, No. 4, pp. 1–9, 2000.
34. T. S. Zhou, G. Chen and Q. G. Yang, A simple time-delay feedback anticontrol method made rigorous, *Chaos*, Vol. 14, pp. 662–668, 2004.
35. H. G. Zhang, Z. L. Wang and D. R. Liu, Chaotifying fuzzy hyperbolic model using adaptive inverse optimal control approach, *Intl. J. of Bifurcation and Chaos*, Vol. 14, No. 10, pp. 3505–3517, 2004.
36. T. Kapitaniak, *Controlling chaos – Theoretical and practical methods in nonlinear dynamics*, Academic Press, 1996.
37. S. Boccaletti, C. Grebogi, Y.-C. Lai, H. Mancini and D. Maza, The control of chaos: theory and applications, *Physics reports*, Vol. 39, pp. 103–197, 2000.
38. K. Tanaka, T. Ikeda and H.O. Wang, A unified approach to controlling chaos via an LMI-based fuzzy control system design, *IEEE Trans. Circ. Syst.—I*, Vol. 45, No. 10, pp. 1021–1040, 1998.
39. H. Wang, K. Tanaka and M. Griffin, Parallel distributed compensation of nonlinear systems by Takagi and Sugeno's fuzzy model, *Proc. FUZZ-IEEE'95*, pp. 531–538, 1995.
40. H. Wang, K. Tanaka and M. Griffin, An approach to fuzzy control of nonlinear systems: stability and design issues, *IEEE Trans. Fuzzy Syst.*, Vol. 4, pp. 14–23, 1996.
41. Z. Li, J. B. Park, G. Chen, Y. H. Joo and Y. H. Choi, Generating chaos via feedback control from a stable TS fuzzy system through a sinusoidal non-linearity, *Intl. J. Bifur. Chaos*, Vol. 12, No. 10, pp. 2283–2291, 2002.
42. Z. Li, J. B. Park, Y. H. Joo, G. Chen and Y. H. Choi, Anticontrol of chaos for discrete TS fuzzy systems, *IEEE Trans. Circ. Syst.-I*, Vol. 49, No. 2, pp. 249–253, 2002.

43. Z. Li, J. B. Park and Y. H. Joo, Chaotifying continuous-time TS fuzzy systems via discretization, *IEEE Trans. Circ. Syst.-I*, Vol. 48, No. 10, pp. 1237–1243, 2001.
44. E. A. Jackson, The entrainment and migration controls of multiple attractor systems, *Phys. Lett. A*, Vol. 151, pp. 478–484, 1990.
45. E. A. Jackson, On the control of complex dynamic systems, *Physica D*, Vol. 50, pp. 341–366, 1991.
46. E. A. Jackson, *Perspectives of nonlinear dynamics*, Vol. 1 and 2, Cambridge University Press, New York, 2nd ed., 1991.
47. Y. Braiman and I. Goldhirsch, Taming chaotic dynamics with weak periodic perturbations, *Phys. Rev. Lett.*, Vol. 66, pp. 2545–2548, 1991.
48. R. Chacón, Geometrical resonance as a chaos eliminating mechanism, *Phys. Rev. Lett.*, Vol. 77, pp. 482–485, 1996.
49. K. Pyragas, Continuous control of chaos by self-controlling feedback, *Phys. Lett. A*, Vol. 170, pp. 421–428, 1992.
50. K. Pyragas, Control of chaos via an unstable delayed feedback controller, *Phys. Rev. Lett.*, Vol. 86, pp. 2265–2268, 2001.
51. J. E. S. Socolar, D. W. Sukow and D. J. Gauthier, Stabilizing unstable periodic orbits in fast dynamical systems, *Phys. Rev. E*, Vol. 50, pp. 3245–3248, 1994.
52. K.-Y. Lian, C.-S. Chiu, T.-S. Chiang and P. Liu, LMI-based fuzzy chaotic synchronization and communications, *IEEE Trans. Fuzzy Systems*, Vol. 9, No. 4, pp. 539–553, 2001.
53. K.-Y. Lian, T.-S. Chiang, P. Liu, and C.-S. Chiu, Synchronization and secure communication for chaotic systems: the modulated fuzzy system design, *Int. J. Bifur. Chaos*, Vol. 11, No. 5, pp. 1397–1410, 2001.
54. K.-Y. Lian, T.-S. Chiang, C.-S. Chiu and P. Liu, Synthesis of fuzzy model-based designs to synchronization and secure communications for chaotic systems, *IEEE Trans. Syst., Man, and Cybern. - Part B: Cybernetics*, Vol. 31, No. 1, pp. 66–83, 2001.
55. K. Short, Steps toward unmasking secure communications, *Int. J. Bifurcation Chaos*, Vol. 4, pp. 959–977, 1994.